



Risk management under the SABR model

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Risk management of interest-rate derivatives can lead to a variety of definitions for each of the Greeks. In this article we examine the differences brought by each of the choices. We also discuss the difference between the Normal and Black calibration spaces.

Introduction

The SABR model owes its popularity to the fact that it can reproduce comparatively well the market-observed volatility smile and that it provides a closed-form formula for the implied volatility. In fact, because of these two features most practitioners use the SABR model mostly as a smile-interpolation tool rather than a pricing tool.

The SABR model is not the first smile-interpolation tool. Market dealers have been trying to fit the smile using a variety of models, ranging from plain statistical regressions across the quoted market points to other semi-rigorous approaches, such as the Vanna-Volga method in FX. Although most of these attempts manage to successfully fit the market smile, they do not provide a self-consistent arbitrage-free solution between pricing vanillas (using the smile) and pricing exotics (using Monte Carlo). On the contrary, the SABR model is setup on a coupled set of stochastic differential equations and can thus, in theory, answer both the need for smile-interpolation and for Monte-Carlo pricing. Theory versus practice, however, can often be miles apart and SABR is not an exception.

The reason is that one of SABR's main strengths is also one of its key weaknesses. The treasured closed-form formula for the implied vol that offers the possibility for quick calibrations and quick quotes is based on an approximation which breaks down for large strikes and large maturities. The fact that this approximation breaks down implies that vanilla versus non-vanilla pricing (analytic formula pricing versus Monte-Carlo pricing) would not be consistent anymore.

The approximation works well for reasonable choices of strikes and maturities offering the possibility to manage and hedge market risks, such as the Delta, Vega, Gamma, etc. In some practices, the daily risk-management goes even further by managing the sensitivities associated to the four SABR parameters.

The SABR formula can be found in two variants: the Black SABR formula or the Normal SABR formula, which, respectively, express Black or Bachelier implied vols in terms of the SABR parameters (this is commonly termed the Black or Normal calibration space).

In recent times, SABR has been brought back to attention due to the negative-rate environment. In this environment a solution is required for quoting volatilities at a negative strike. Currently, two mainstream approaches have emerged: (i) the classical SABR model in the Black calibration space is modified by introducing a shift and (ii) the classical SABR model is maintained but it is only used in the special case of $\beta=0$ and under the Normal calibration space.

In this article we provide some indications regarding the use of SABR in daily risk-management under either the Black or the Normal variant.

Calibration space

The SABR model expresses the implied volatility either in terms of a Black volatility (which will be input to a Black'76 formula) or in terms of a Normal volatility (which will be input to a Bachelier formula). In recent years, with the interest-rates going into the negative domain there has been an obvious obstacle in any Black pricing engine: the Black formula requires the computation of the logarithm of the forward and of the strike, which, if negative, leads to an unpleasant exception error.

One quick fix to this problem is to shift both the forward and the strike so that the logarithm is not undefined. This then leads to a new model, the *shifted* SABR model. The asymptotic expansions of Hagan et. al. and Berestycki et. al. can easily be adapted in order to deal with this shift. In a Black world, these formulas would no longer yield the common Black volatility, rather, they quote a shifted Black volatility. In the Normal world, the Hagan asymptotic expansion will yield a Normal volatility (note that in the Bachelier formula, the shift on the strike and the shift on the forward will cancel each other out, meaning that there is no impact of a shift on a normal model). Calibration of the SABR model using a shifted Black SABR asymptotic formula would be called the "(shifted) Black calibration space", whereas a calibration using the Normal asymptotic formula is called the "Normal calibration space".

Black Calibration Space

$$\sigma_B(K, f) = \frac{\alpha}{(f\beta)^{\frac{(1-\beta)}{2}} \left\{ 1 + \frac{(1-\beta)^2}{24} \log^2 \frac{f}{K} + \frac{(1-\beta)^4}{1920} \log^4 \frac{f}{K} + \dots \right\}} \cdot \left(\frac{z}{\chi(z)} \right) \cdot \left\{ 1 + \left[\frac{(1-\beta)^2}{24} \frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4} \frac{\beta\rho\alpha v}{(fK)^{\frac{(1-\beta)}{2}}} + \frac{2-3\rho^2}{24} v^2 \right] \right\} \cdot T + \dots$$

where z and $\chi(z)$ are abbreviations of

$$z = \frac{v}{\alpha} (fK)^{\frac{(1-\beta)}{2}} \log \frac{f}{K}$$

$$\chi(z) = \log \left(\frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1 - \rho} \right)$$

Normal Calibration Space

$$\sigma_N(K, f) = \alpha (fK)^{\frac{\beta}{2}} \frac{1 + \frac{1}{24} \log^2 \frac{f}{K} + \frac{1}{1920} \log^4 \frac{f}{K} + \dots}{1 + \frac{(1-\beta)^2}{24} \log^2 \frac{f}{K} + \frac{(1-\beta)^4}{1920} \log^4 \frac{f}{K} + \dots} \cdot \left(\frac{z}{\chi(z)} \right) \cdot \left\{ 1 + \left[\frac{-\beta(2-\beta)}{24} \frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4} \frac{\beta\rho\alpha v}{(fK)^{\frac{(1-\beta)}{2}}} + \frac{2-3\rho^2}{24} v^2 \right] \right\} \cdot T + \dots$$

These expressions hold for any value of beta. Notice that a shift is necessary (due to the presence of logarithms of the strike and forward) in either the Black or the Normal calibration space.

The *Hagan et al* article derives in the special case of beta=0 a convenient expression for the normal implied vol¹. The expression in the Normal beta=0 case contains no logarithms and can thus be used, in the presence of negative strikes or forwards, without the need to introduce a shift. This implies that there is no need for laborious software adaptations apart from fixing the parameter beta to the value of zero, provided the pricing library can already handle a calibration in the Normal space with beta=0.

There is a certain amount of discussion in the literature and among market dealers of whether a SABR model with beta=0 would be an appropriate model. This is because setting beta equal to zero in the SABR model would lead to a stochastic differential equation for the forward whereby the increments to the forward rate do not depend on its current value. This is, from a phenomenological point of view, a problematic issue, although, from a pricing point of view, all that matters to a model is its calibration to the observed market prices.

¹ The reader is warned that the normal beta=0 formula in the *Hagan et al* paper contains a typo error.

Calibration workflow

The main steps in a typical workflow in the calibration of the cap market are:

Market Data

Market data can be provided either in the format of Normal Cap Vols or in Black Cap Vols. Normal Cap Vols can be found quoted for negative strikes or forwards, while Black cap vols (without a shift) can be found only for positive strikes or forwards. A significant component of the market data is the liquidity of the quotes. Typically, apart from quotes in the neighbourhood of the ATM point, liquidity is limited and our trust in the data quality is a leap of faith.

Cap Prices

Once market vols are obtained they are converted to cap prices via the Black or Bachelier formulas. This conversion is necessary because the SABR model calibrates on caplet vols -not cap vols.

Caplet Vols

Caplet vols need to be stripped from the cap prices. The result can be expressed either in shifted Black caplet vols or in normal caplet vols (and this choice does not depend on whether market data are in Normal or Black format). It is at this stage that one has to make a choice of shift, when using a Black calibration space.

Calibrate SABR

To calibrate the SABR model one of the two asymptotic formulas need to be used: Normal or Black. This choice is linked to the choice of how to express the caplet vols. It is commonly called calibration space. It is at this stage that one has to make a choice of shift when using a Normal calibration space.

Express MTM as cap vol

Once the MTM has been obtained it is efficient to re-express it in terms of a vol, which can be a shifted Black cap vol or a Normal cap vol.

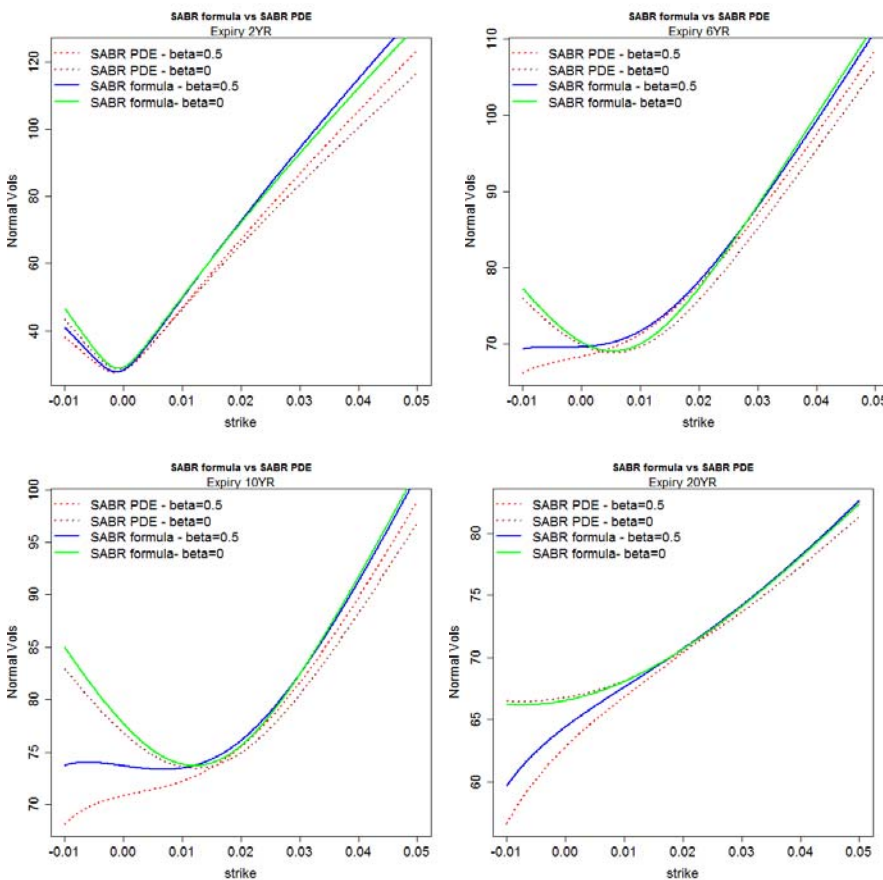
At various steps in this procedure one finds a choice between "Normal or Black" which if quoted without the appropriate context can be confusing as to what it refers.

Black vs Normal calibration space

In the special case of $\beta=0$ the Normal asymptotic formula of the SABR model offers the convenient possibility to price derivatives with negative strikes without introducing a shift. This is an appealing feature of the model, as no adjustments to the pricing infrastructure are needed, provided a code for computations on the Normal space is already in the pricing library.

It is reasonable then to examine how the $\beta=0$ Normal model compares to the (more standard) $\beta=1/2$ shifted Black model.

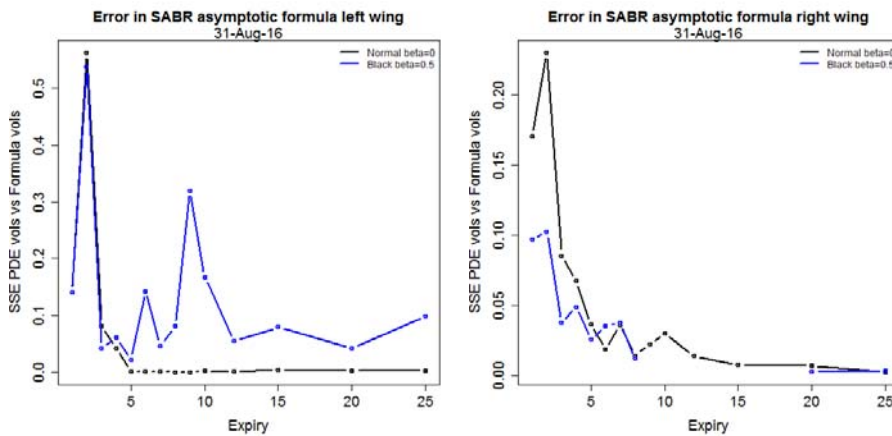
In order to do that we have taken as a reference the solution of the SABR PDE, which does not rely on the approximation formula. We have priced, under both models, vanilla caps of different expiries and compared the resulting caplet volatility smiles. In the below figures we illustrate the result for the expiries of 2Y, 6Y, 10Y and 20Y. Our test case is based on calibrating the EUR6M cap market as of 31 August 2016. The shift in the shifted Black model is taken as 2%.



Notice the emergence of the “SABR knee” (last figure) for the beta=0.5 Black model. This feature is not present in these figures in the Normal model.

Compared to their corresponding PDE solutions (dotted lines) both model behave similarly, although the approximation breaks down for large strikes and large maturities.

The two models have a somewhat different wing behavior for short maturites. This can be seen in the following figures that show the sum of square errors between SABR-formula vols and PDE-vols for various expiries. The sum of square errors runs across strikes (left of the forward or right of the forward).



As a conclusion, one could say that the approximation error in both models is roughly equivalent.

We emphasise that the graphs above describe the difference between the Hagan/Berestycki asymptotic expansions versus the SABR PDE solution (for the same Alpha, Beta, Rho and Nu). This difference is not an error in calibration, rather, it reflects the breaking down of the asymptotic expansion. When calibrating SABR parameters to the market using the asymptotic expansions, we remain internally consistent provided we also use the same expansion when pricing. The difference described above will only become relevant when using the asymptotic expansion for calibration purposes, and subsequently using the full PDE/SDE for pricing.

The Delta sensitivity

In interest-rate derivatives, the Delta refers to the sensitivity of the price of the derivative with respect to a small movement of the underlying forward rate.

Among the various inputs that the SABR formula expects from the user is the forward rate. This implies that shifting the forward rate in order to compute the Delta would modify the SABR volatilities. If the practitioner requires that the effect of the forward rate shift is isolated, then the volatilities need to be forced to remain unmodified. In this case, there is a further choice to make, concerning the type of vols that should remain the same. In other words, should these be the cap vols or the caplet vols? Both of these are valid options for risk-management purposes although they would give different results and the user needs to be aware of what is bumped (and also of what is not).

Choosing to maintain the same cap vols during a shift of the forward curve would mean that the term structure of the caplet vols would have to be modified if the forward rate is shifted. This is because the cap volatility is an artificial vol that converts the value of the derivatives in terms of a volatility. Therefore, shifting the forward curve, would imply a different price and hence a different series of caplet vols.

A final choice refers to the mode of pricing. There are two market-standard ways of computing caplets: either using a shifted Black formula or a Bachelier formula. In the former case the input is a Black vol while in the latter case the input is a normal vol. These formulas behave differently when the forward rate is modified, even when they lead to exactly the same MTM.

Conveniently, once the SABR model is calibrated, the Black caplet vols and the Normal caplet vols can be generated from the SABR asymptotic formulas using the same SABR parameters.

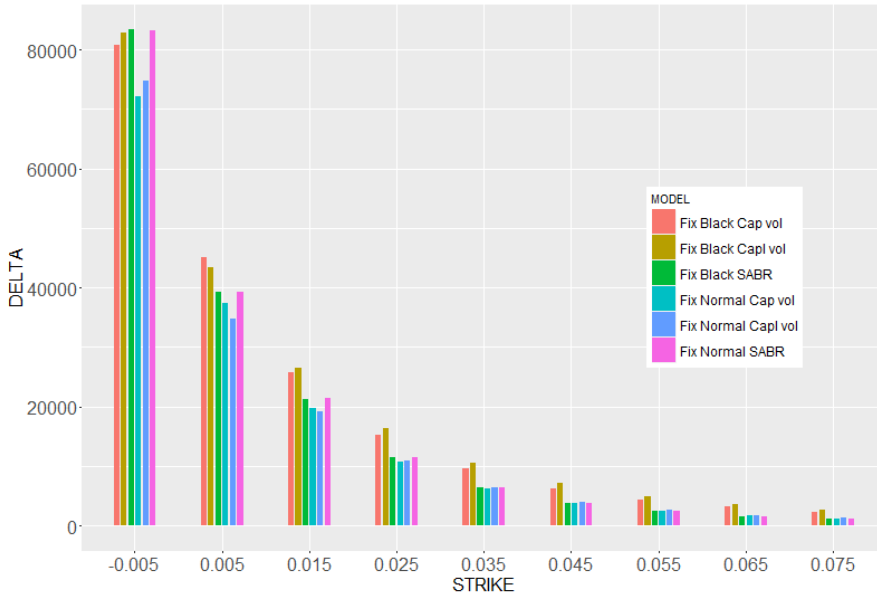
Below are some possible choices for the computation of the Delta:

Possibilities for a Delta computation

- Shift the Forward curve
 - Shift the Forward and the Discount curve
-
- During the curve shift, keep the same cap vols
 - During the curve shift, keep the same caplet vols
-
- Price caplets using Black vols and the shifted Black formula
 - Price caplets using Normal vols and the Bachelier formula
-

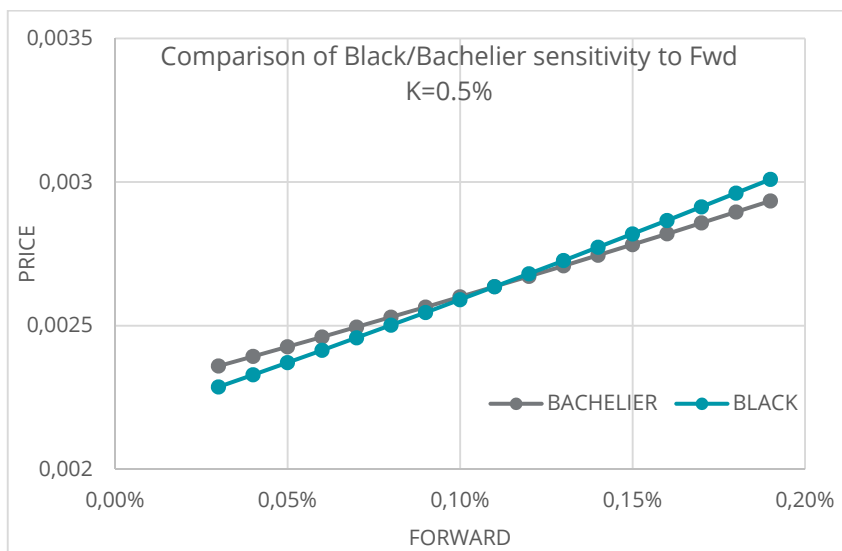
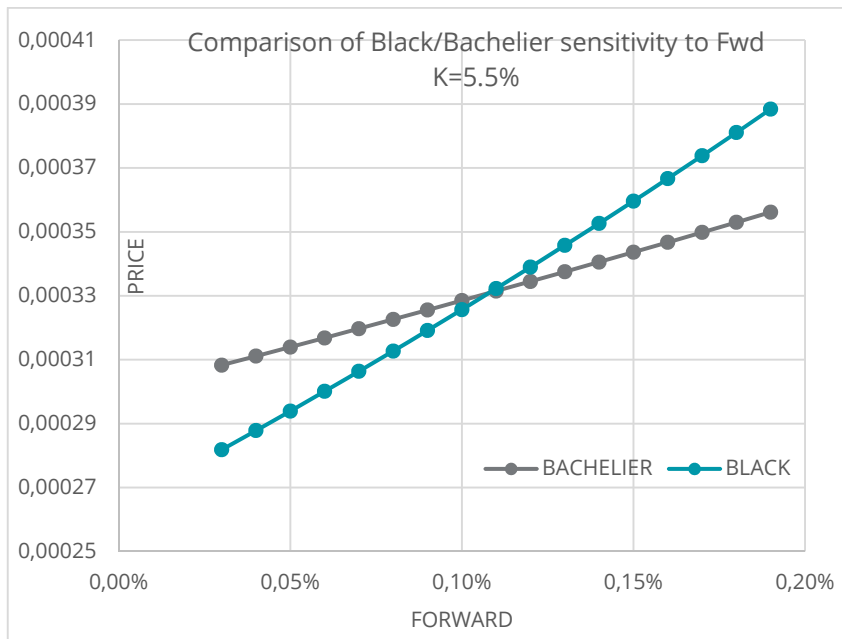
All of these choices can be made independently of one another, which means that, in this case there would 8 possible Deltas.

In the figure below we illustrate some of these possible Deltas. The horizontal axis represents the strike while the vertical axis represents the Delta. This figure is obtained by calibration to the EUR 6M cap market as of 31 August 2016.



In the figure above we see how the Deltas obtained via fixing cap vols are not the same as those obtained via fixing caplet vols. The magnitude of the difference depends on the term-structure of the caplet vols. In particular, we anticipate that since most of cap value lies in the last caplets of the cap, the difference between the last caplet vols and the cap vol would also control the difference between the two Deltas.

We also notice that fixing the SABR parameters leads to similar Deltas regardless of whether we are pricing in the Normal space or the Black space. This again makes good sense in view of the fact that the same SABR parameters lead to the same price regardless of whether one prices in the Black or Normal space. Finally we see that Black Deltas are somewhat larger than Normal Deltas. This is due to the different reaction of the two formulas under a curve shift. To provide some further intuition regarding this behaviour we compare, in the figure below, the sensitivity of the Black versus the Bachelier formulas to the forward rate for two different strikes.



In both figures the slope of the price versus forward is larger in the Black space.

Slope $\Delta(MTM)/\Delta F$	K=0.5%	K=5.5%
BLACK	0.45	0.07
BACHELIER	0.36	0.03

The Vega sensitivity

The Vega represents the sensitivity of the price of a derivative with respect to a shift in the volatility. There are various different volatilities that enter into the computation and a shift in either of them could be a valid choice for the Vega. In more detail, some of the possibilities are the following:

Possibilities for a Vega computation

- Shift the cap vols
 - Shift the caplet vols
 - Shift the input market vols
-
- Price caplets using Black vols and the shifted Black formula
 - Price caplets using Normal vols and the Bachelier formula
-

The various choices for shifting the vols represent the following shift:

- **Shift the cap vol:** In this case a shift is made on the cap vol generated by SABR that yields the MTM. This does not require a re-calibration of the SABR parameters.
- **Shift the caplet vols:** In this case a shift is made on each of the caplet vols produced by SABR that price the cap. This does not require a re-calibration of the SABR parameters.
- **Shift the input market vols:** In this case a shift is made on each of the market data cap vols as taken from the data provider. This does require a re-calibration of the SABR parameters.

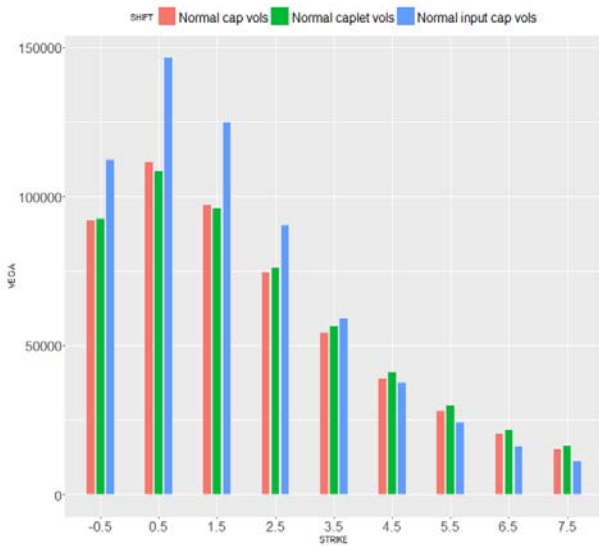
According to market standards the shift is typically 1% for Black volatilities while 0.01% for Normal volatilities. In some cases, users prefer to use smaller shifts, in order to comply with the requirement that the movement should be small, and rescale the MTM accordingly at the end of the computation.

Across the three possible shifts quoted above it would be tempting to see the third Vega, the result of shifting the input market data, as the appropriate way of computing the Vega, as it is based on shifting an observable quantity. However, this is entirely a matter of choice, and no choice is better than any other as long as one hedges one's positions accordingly. Notice, however, that under a shift of the input market data, a full re-calibration of the SABR parameters is required. This is not the case for the first two ways of computing the Vega, based on shifting cap or caplet vols. Market practice often disregards choices that require compromises in computational efficiency.

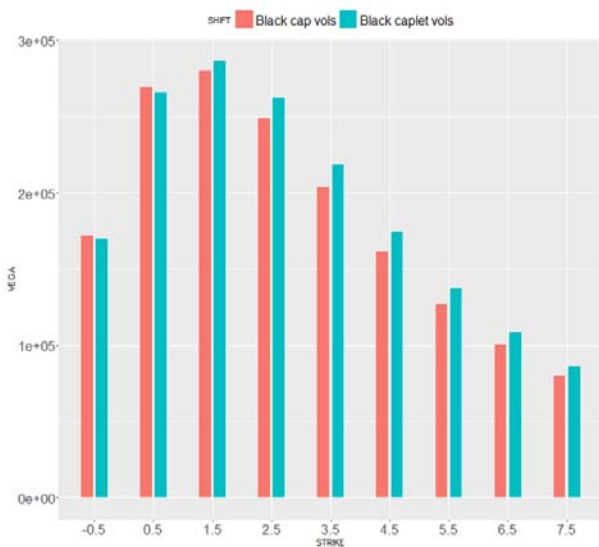
A further choice regarding the Vega concerns the type of formula used. One can price caplets using either Black or Bachelier formulas and a shift in vols in either would generate different values.

In the figures below we show the value of Vegas for a variety of scenarios. These have been generated by calibration to the EUR 6M cap market as of 31 August 2016 using the 12Y expiry.

The first figure below shows the values of the Vegas in the Normal space. Three Vegas have been considered, corresponding to shifts in cap vols, caplet vols or input cap-vol market-data. In this test case the market data was expressed in terms of Normal vols which makes the comparison across the three cases sensible. Interestingly, similar results are obtained by a shift in cap vols or caplet vols.



The figure below shows a similar computation but in a Black space. Notice that the Black Vegas are on a completely different scale than the Normal Vegas. This is due to the fact that they represent different volatilities: The Black volatility represents a relative increment to the forward, while the Normal volatility represents an absolute increment.



In practice we note that the Vega is not often a quantity used in Front-Office risk management. Banks using a (shifted) SABR would hedge the individual SABR parameters, rather than the traditional Black/Bachelier Vega. A Vega, however, will still have to be calculated for regulatory purposes (in particular the Fundamental Review of the Trading Book).

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