Calibration and pricing using the free SABR model

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Contents

This article looks into some of the features of the free SABR model, in particular in the context of the negative-rate environment.

Introduction

The SABR model has become the dominant tool for smile-interpolations in the interest-rate world owing to two distinct features: Firstly, the fact it is a stochastic volatility model and can therefore fit the volatility smile, and, secondly, the fact that it allows for an approximate closed-form formula that expresses the implied volatility (Black or Bachelier) in terms of the model's parameters.

Under the negative-rate environment the SABR model as well as the traditional Black model cannot work. The reason is that by-design the models expect forwards and strikes to be strictly positive. An exception error appears whenever this is the case.

In order to circumvent this exception problem the market has introduced a "shift" in both the forward rate and the strike. The shift is such that, when added to the strike and the forward, the relevant mathematical quantities remain well-defined.

An alternative approach to handle pricing of interest-rate derivatives in the negative-rate environment is the introduction of new models that can by-design handle negative rates. One such approach is the free SABR model by Antonov et al.[1].

In this article we examine some of the features of this model and investigate its similarities to the traditional SABR model.

[1] A Antonov, M Konikov and M Spector, "The Free Boundary SABR: Natural Extension to Negative Rates", available at ssrn.com

Model description

The free SABR model can be seen as a natural extension of the classical SABR model. The main strength of this model is that it is designed to be able to handle the possibility that the forward rate can become negative. This is done in a simple and elegant fashion by introducing the operator of the absolute value in the relevant stochastic differential equation of the forward:

	Classic SABR	Free SABR
Stochastic Differential Equation	$dF_t = \alpha \cdot {F_t}^\beta \cdot dW_t$ $d\alpha = v \cdot \alpha \cdot dZ_t$ $E^{Q_T}[dW_t \cdot dZ_t] = \varrho dt$	$dF_t = \alpha \cdot \lvert F_t \rvert^\beta \cdot dW_t$ $d\alpha = v \cdot \alpha \cdot dZ_t$ $E^{Q_T}[dW_t \cdot dZ_t] = \varrho dt$
Range of F_t	The forward can only be positive	The forward can be positive or negative
Analytical solution	No general exact analytical solution exists, yet many analytical approximations have been derived (by Hagan *et al*, Berestycki *et al*, Henri-Labordele,etc.[2]).	No general exact analytical solution exists (yet there is one for the Rho=0 case). Analytical approximations have been developped using a Markovian projection to the Rho=0 case.

In terms of the stochastic differential equations, the free SABR model differs from the classic SABR only in the presence of an absolute value that operates on the current forward value (right-hand side of the SDE) to give the new increment (left-hand side).

The advantage of injecting an absolute value can be best thought of in terms of a Monte-Carlo thought-simulation of the above SDE: Regardless of the value of the current sign of the forward F_t (positive of negative) the new increment dF_t is always well-defined. Its sign depends on the sign of the parameter α and the value of the Gaussian random variable dW_t. But there is no value of the combination of (α, dW_t, F_t) that can lead to an exception. On the contrary, in the classic SABR model for any $\beta > 0$ the current value of the forward F_t is required to be positive (e.g. for $\beta = \frac{1}{2}$ one is required to compute $\sqrt{F_t}$ before obtaining the value of the increment for this time step).

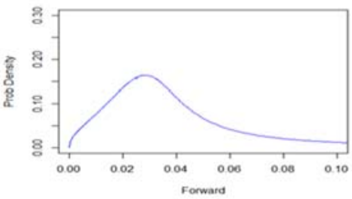
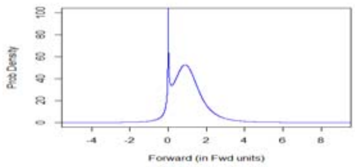
From this respect, the free SABR model is a simple and elegant extension to the classic SABR model: Without changing much in the defining relations of the model, the entire landscape of possibilities has changed.

[2] Hagan et al "Managing Smile Risk" Wilmott Magazine (7/2002),
Berestycki *et al.* "Computing the implied volatility in stochastic volatility models" Comm. Pure Appl. Math., 57 1352, 2004,
Pierre Henry-Labordele "A general asymptotic implied volatility for stochastic volatility models", available at arxiv.org and ssrn.com

Singularity and stickiness

The introduction of the absolute value in the SDE of the free SABR model gives rise to some interesting dynamics.

In particular, the probability density function of the forward shows two maxima, one of which occurs at exactly F_t=0. This singularity at F_t=0 corresponds to a high probability that the forward attains the value. A heuristic reason for this can be given, again, in terms of a Monte-Carlo thought-simulation: For those simulation paths of the forward that approach the value F_t=0 the stochastic differential equation dictates that the value of the next increment dF_t is (as proportional to the current value) close to zero. As a result, these paths cannot move significantly away from zero and in fact get trapped around this value.

	Classic SABR / shifted SABR	Free SABR
Singularity		
Stickiness	There is no singularity at 0, and hence no special "stickiness" near a zero forward rate.	The singularity at the zero forward, has the result that zero acts as an attractor of Monte-Carlo paths. Once the forward rate is close to zero, its deviations will be small (as they are proportional to |F|). This phenomenon is called the stickiness at zero.
PDF	A smooth probability distribution of the forward rate	A singularity in the probability distribution function at zero.

This stickiness of the forward to the value of zero has been observed in recent times. For example, the CHF 3M and 6M forwards have remained very close to zero since the beginning of the financial crisis of 2008. Some questions do remain however. For example, is this stickiness something fundamental to the market or just a short-term observation? If at some future point the CHF forward moves away from zero can we still say that the free SABR model describes well the market movements?

Methods of solution

In this section we outline the two main ways of solving the free SABR model and obtain a derivative's price.

This PDE can be solved using various numerical techniques, as described in the article "Finite Difference Techniques for Arbitrage-Free SABR" by F. Le Floc'h and G. Kennedy (article available in ssrn.com).

Once the probability density function is known an option payoff can be easily priced using either integration or Monte-Carlo techniques).

PDE exact solution

The probability density function $Q(T,F)$ of the forward rate F at a future time T is described by the second-order partial differential equation[3]

$$\frac{\partial Q}{\partial T}(T,F) = \frac{\partial^2 M(T,F)Q(T,F)}{\partial F^2}$$

with

$$M(T,F) = \frac{1}{2}D^2(F)E(T,F)$$

$E(T,F) = e^{\rho\nu\alpha\Gamma(F)T}$, $\Gamma(F) = \frac{F^\beta - f^\beta}{F-f}$, $D(F) = \sqrt{\alpha^2 + 2\alpha\varrho\nu y(F) + \nu^2 y(F)^2}F^\beta$, $y(F) = \frac{F^{1-\beta} - f^{1-\beta}}{F-\beta}$

and initial condition

$$\lim_{T\to 0} Q(T,F) = \delta(F-f)$$

Closed-form formulas

As is the case for the (shifted) SABR, there exist asymptotic expansions for the Free Boundary SABR. In fact, there is a closed form exact solution for the time value $O_F^{SABR}(T,K)$ of a call option, in the zero correlation (ρ=0) case. From the Antonov *et al* article "The Free Boundary SABR: Natural Extension to Negative Rates" we quote the solution for the time value[4]:

$$O_F^{SABR}(T,K) = \mathrm{E}[(F_T - K)^+] - (F_0 - K)^+ = \frac{1}{\pi}\sqrt{|KF_0|}\{1_{K\geq 0}A_1 + \sin(|\nu|\pi)A_2\}$$

Where

$$A_1 = \int_0^\pi d\phi\, \frac{\sin\phi\ \sin(|\nu|\phi)\, G\left(T\gamma^2, s(\phi)\right)}{(b-\cos\phi)\cosh s(\phi)}$$

[3] See "Arbitrage Free SABR" by Patrick S Hagan, Deep Kumar, Andrew Lesniewski and Diana Woodward, Wilmott (2014) 69

[4] We refer the reader to the Antonov *et al* article for the details behind the derivation of the formulas.

$$A_2 = \int_0^\infty d\psi \; \frac{\sinh\psi\,(1_{K\geq 0}\cosh(|\nu|\psi) + 1_{K<0}\sinh(|\nu|\psi))}{b+\cosh\psi}\frac{G\big(T\gamma^2, s(\psi)\big)}{\cosh s(\psi)}$$

With

$$G(t,s) = 2\sqrt{2}\frac{e^{-\frac{t}{8}}}{t\sqrt{2\pi t}}\int_s^\infty du\, u\, e^{-\frac{u^2}{2t}}\sqrt{\cosh u - \cosh s}$$

$$\sinh s(\phi) = \gamma\alpha^{-1}\sqrt{2\bar{q}(b-\cos\phi)}$$

$$\sinh s(\psi) = \gamma\alpha^{-1}\sqrt{2\bar{q}(b+\cosh\psi)}$$

$$\bar{q} = q_0 q_K \;\; b = \frac{q_0^2+q_K^2}{2q_0q_K} \;\; q_0 = \frac{|F_0|^{1-\beta}}{1-\beta} \;\; q_K = \frac{|K|^{1-\beta}}{1-\beta} \;\; \nu = -\frac{1}{2(1-\beta)}$$

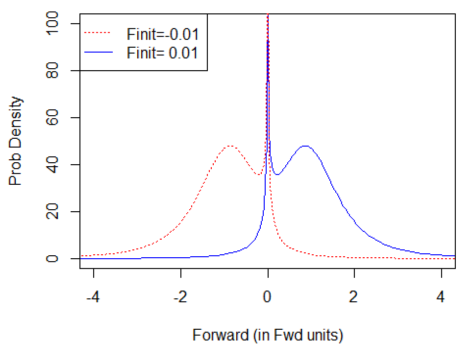
From the above equations we see that the above solution involves the computation of a double integral. This can be a delicate and time-consuming operation. The article “SABR spreads its wings” (2013) by A Antonov, M Konikov and M Spector *Risk* 26.8 (58) derives an asymptotic expansion of the function $G(t,s)$ which reduces the overall computation of O_F^{SABR} to a single integral.

For the general correlation case, $\rho \neq 0$, the Free Boundary SABR article generates an asymptotic solution by means of a projection onto the zero correlation case. That is, a projection occurs from the SABR parameters α, β, ρ, ν onto $\bar{\alpha}, \bar{\beta}, \bar{\nu}$ such that $O_F^{SABR}(T,K,\alpha,\beta,\varrho,\nu) = O_F^{SABR}\big(T,K,\bar{\alpha},\bar{\beta},0,\bar{\nu}\big)$. For further details, see the article “SABR spreads its wings” (referenced in the paragraph above).

Numerical testing

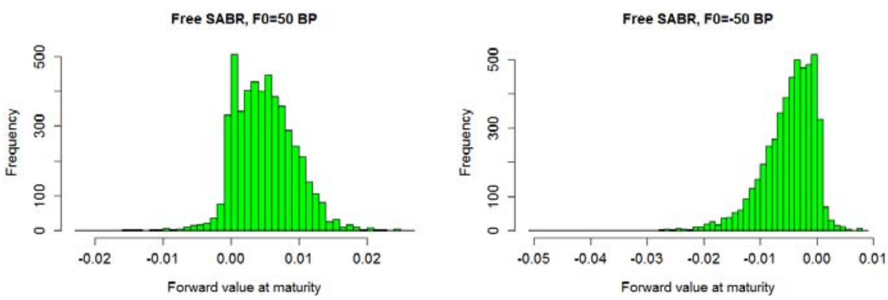
Impact of initial forward value in the probability density function

From the SDE one can already see that the initial sign of the forward will impact the forward interest rate paths. This is because the singularity at zero implies that those forward rate paths starting with a negative value F_{init} <0 will predominantly remain in the negative domain, whereas the converse will hold for positive initial conditions F_{init} >0. In the following figures we plot the probability density function for the scenarios F_{init} = 0.01 and F_{init} = -0.01:



In this figure above we notice the dependence of the PDF on the initial conditions. Although in both cases of F_{init} >0 and F_{init} <0 the PDF carries the two-peak structure, the weight of the probability density has moved abruptly once the initial forward value switched sign. In this case we have used the following parameters: shift=0, $\beta = 0.25$, $\nu = 0.3$, $\rho = 0$, $T = 10$ and $\alpha = 0.3\alpha|F|^{1-\beta}$.

This observation is also seen when carrying out Monte–Carlo simulations of the free SABR model. The figures below show the histogram of the resulting frequencies of forward values at maturities for positive (left) versus negative (right) initial conditions.



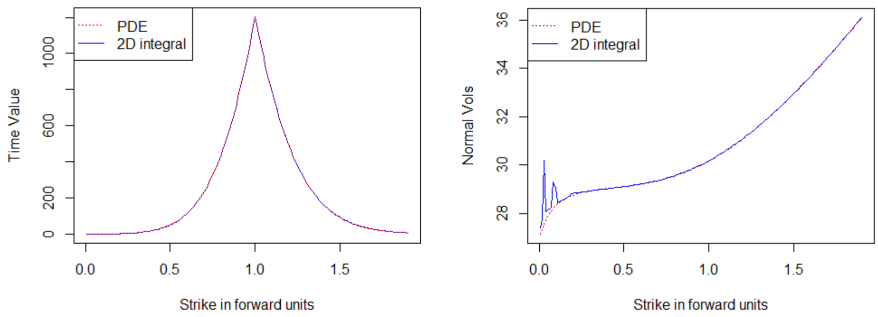
The singularity at zero implies:

- F_{init} cannot be set to equal to zero (as this yields the solution F=0 for all t). F_{init} close to zero will give difficulties in the calibration, as the singularity at zero becomes more prevalent, and becomes harder to capture in the numerics.
- Given a fixed set of α, β, ρ, ν changing from $F_{init}>0$ to $F_{init}<0$ will yield completely different option values. We anticipate that this will impact the "calibration stability" of the SABR parameters across different tenors/expiries and across different valuation dates.

Asymptotic solution versus PDE solution

In order to examine the asymptotic formulas derived by Antonov *et al* we obtain the value of the call option using 2 methodologies:

- Soving the PDE
- Using the analytical formula (2D integral)



We obtain a great agreement in terms of option price. For small strikes we see that the integral solution becomes slightly unstable (yet note that at these levels the option will be equal almost its intrinsic value). This can be mainly due to the issues brought by the numerical integration recipe. For this plot we have used the standard R package for integration. Note that more tailored numerical recipes would resolve these instabilities.

We remark that we have found that the computation time of the PDE solution is faster than the analytical solutions (even when the 1D integral approximation is made). For this reason, an efficient PDE solving scheme might be a better solution.

Advantages of the free SABR model

The free SABR model has a number of appealing features. For example, it captures the “stickiness” features that has been observed in the CHF market and this stickiness can be customised by introducing a shift to the model. Furthermore, the model carries the same number of parameters as the classic SABR model and is therefore able to reproduce a variety of smiles.

Another great advantage is the existence of a closed-form formula in the special case of zero correlation. This circumvents the problem of solving the 2D integral. Finally it is a simple extension of the classic SABR model that can, in an elegant way, handle negative interest rates.

As a disadvantage, the numerical calibration is not as efficient as the classic (shifted) SABR model and the user is likely to observe an instability of the calibrated parameters when the initial conditions of the forward move from the positive domain to the negative domain.

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Contacts

Nikos Skantzos
Director
Diegem

T: +32 2 800 2421
M: + 32 474 89 52 46
E: nskantzos@deloitte.com

Kris Van Dooren
Senior Manager
Diegem

T: +32 2 800 2495
M: + 32 471 12 78 81
E: kvandooren@deloitte.com

George Garston
Senior Consultant
Zurich (Switzerland)

T: +41 58 279 7199
E: gggarston@deloitte.ch